\# 1. (20 pts) Can the limit $\lim_{\varepsilon\to 0+} \varepsilon^{12-N} \int_{E_\varepsilon} |x|^{-12} (\cos x_1)\, dx$ exist? Find the limit if the answer is yes. Here $N \geq 3$, $E_\varepsilon = \{x = (x_1, \cdots, x_N) \in \mathbb{R}^N : |x| \geq \varepsilon\}$, and $|x| = \left(\sum_{j=1}^{N} x_j^2\right)^{1/2}$. Prove or disprove your answer.

\# 2. (20 pts) Let H be a metric space composed of continuous functions on $[0,1]$ with the metric defined by $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ for $f, g \in H$. Let $M_0 = \{h \in H : \|h\|_1 < 1\}$, $M_1 = \{f \in H : \|f\|_1 \leq 1\}$, $M_2 = \{g \in H : \|g\|_2 \leq 1\}$ and $M_3 = \{u \in H : \|u\|_2 > 1\}$, where $\|f\|_1 = \max_{x\in[0,1]} |f(x)|$ and $\|g\|_2 = \|g\|_1 + \sup_{x,y\in[0,1], x\neq y} \frac{|g(x) - g(y)|}{|x-y|}$.

(1) Can M_0 be open in H? (5 pts)

(2) Can M_1 be closed in H? (5 pts)

(3) Can M_2 be closed in H? (5 pts)

(4) Can M_3 be open in H? (5 pts)

Prove or disprove all your answers.

\# 3. (20 pts) Let $\|a\|_1 = \sum_{n=1}^{\infty} e^{-n} |a_n|$, $\|a\|_2 = \sum_{n=1}^{\infty} e^{n} |a_n|$ and $\|a\|_3 = \sup_{n\in\mathbb{N}} |a_n|$ for $a = \{a_n\}_{n=1}^{\infty}$, where each $a_n \in \mathbb{R}$. Must $\|\cdot\|_j, j = 1,2,3$ be equivalent? Here "$\|\cdot\|_i$ is equivalent to $\|\cdot\|_k$" means that there exists a positive constant C independent of a such that $\|a\|_i \leq C\|a\|_k$ for $i \neq k$. Prove or disprove your answer.

\# 4. (20 pts) Let $\phi : [-1,1] \to \mathbb{R}$ be a continuous function. Must the following inequality $\dfrac{\int_{-1}^{1} \phi\, e^{\phi}\, dx}{\int_{-1}^{1} e^{\phi}\, dx} \geq \dfrac{\int_{-1}^{1} \phi\, e^{-\phi}\, dx}{\int_{-1}^{1} e^{-\phi}\, dx}$ hold? Prove or disprove your answer.

\# 5. (20 pts) For $p > 0$, let $S_p = \{(x,y) \in \mathbb{R}^2 : |x|^p + |y|^p = 1\}$.

(1) Must S_p be a smooth curve for $p = 2n, n \in \mathbb{N}$ (i.e. n is positive integer) ? (10 pts)

(2) Must $S_{1/2}$ (i.e. $p = 1/2$) be a smooth curve? (10 pts)

Prove or disprove all your answers.