- In the following "function" means "$\mathbf{R}$-valued function."
- In a metric space, we let $B_r(p)$ denote the open ball with center p and radius r.

一、是非題

(每題 2 分。答案卷上請標明題號並依序以 ◯/× 分別表示「是/非」作答。)

1. If I_n ($n \in \mathbf{N}$) are bounded open intervals in $\mathbf{R}$ such that $\emptyset \neq I_{n+1} \subseteq I_n$ for all n, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.
2. A continuous convex function on $[-1, 1]$ must be a Lipschitz function.
3. There exists a continuous function f on $\mathbf{R}$ such that $f'(x)$ exists if and only if $x \neq 0$.
4. There exists a continuous function f on $\mathbf{R}$ such that $f'(x)$ exists if and only if $x = 0$.
5. If $V(x, y) = (P(x, y), Q(x, y))$ is a smooth vector field on an open set Ω in $\mathbf{R}^2$ such that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on Ω, then the line integrals of V along any two smooth paths in Ω both travelling from a given point a to another point b coincide.
6. If f is a continuous function of $\mathbf{R}$ of period 2π, then the Fourier series of f converges pointwise to f.
7. If f is a smooth map from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that the determinant of the Jacobi matrix of f takes value 0 at a point $p \in \mathbf{R}^2$, then $f|_{B_r(p)}$ cannot be injective no matter how small $r (> 0)$ is.
8. Let f be a smooth function on $\mathbf{R}^2$ which achieves a local minimum at a point (a, b). If (a, b) is the *only* point at which the gradient vector ∇f vanishes, then $f(a, b)$ must be the global minimum of f.
9. If f and g are functions on $[0, 1]$ such that both f^2 and g^2 are Riemann integrable, then $(f + g)^2$ is Riemann integrable.
10. $\lim_{x \to 0^+} x^x = 0$.

二、計算與證明

(請在答案卷上標明題號，作答時不需依照題目編號順序。注意！時間短暫。)

1.(15 分) Let $a > 1$ and $b > 0$. How many different $x \in \mathbf{R}$ can fulfil the equation $a^x = |x|^b$? (The answer depends on the relation between a and b, and your answer has to exhaust all possibilities.)

2.(10 分) Let f be a function defined on a subset X of $\mathbf{R}^2$. Prove the following statement: if $f|_B$ is uniformly continuous for every bounded set $B \subseteq X$, then there exists a continuous function g on the closure $\bar{X}$ of X in $\mathbf{R}^2$ such that $f = g|_X$.

見背面

3.(10 分) Show that if f is a *uniformly* continuous function on $[0,\infty)$ such that the improper integral $\int_0^\infty f(x)dx$ converges, then $\lim_{x\to\infty} f(x) = 0$.

4.(10 分) Let $C = \{(x,y) \in \mathbf{R}^2 | x^3 = y^2\}$. Show that for any continuously differentiable map $t \in \mathbf{R} \longmapsto (x(t), y(t)) \in \mathbf{R}^2$ whose image lies in C such that $(x(0), y(0)) = (0,0)$ we must have $(x'(0), y'(0)) = (0,0)$.

5.(10 分) **Terminology.** Let f_n be a sequence of functions on a metric space X. We say that f_n converges *compactly* to a function g defined on X if on every compact set K in X the sequence $f_n|_K$ converges to $g|_K$ uniformly.

Now suppose that f_n is a sequence of continuously differentiable functions on an open set U in $\mathbf{R}^2$ such that the three sequences f_n, $\frac{\partial f_n}{\partial x}$, and $\frac{\partial f_n}{\partial y}$ converge compactly to functions g, h_1, and h_2, respectively. Show that $\frac{\partial g}{\partial x} = h_1$ and $\frac{\partial g}{\partial y} = h_2$.

6.(10 分) Let f be a function on an open set U in $\mathbf{R}^2$ such that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere on U. Show that if $\frac{\partial f}{\partial x}$ is continuous at a point $(a,b) \in U$, then

$$\lim_{(x,y)\to(a,b)} \frac{\left|f(x,y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)(x-a) - \frac{\partial f}{\partial y}(a,b)(y-b)\right|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

7. **Terminology.** For a bounded function f on an interval $[a,b]$ $(a < b)$ and a subdivision $\Delta : a = x_0 < \cdots < x_k = b$ of $[a,b]$, we define the upper sum and the lower sum of f with respect to Δ by

$$\overline{S}(f,\Delta) := \sum_{j=1}^{k} \sup_{x_{j-1}\leqslant x\leqslant x_j} f(x)(x_j - x_{j-1})$$

and

$$\underline{S}(f,\Delta) := \sum_{j=1}^{k} \inf_{x_{j-1}\leqslant x\leqslant x_j} f(x)(x_j - x_{j-1}),$$

respectively. A function f on $[a,b]$ is called Darboux integrable if it is bounded and for any given $\varepsilon > 0$ there exists a subdivision Δ of $[a,b]$ such that $\overline{S}(f,\Delta) - \underline{S}(f,\Delta) < \varepsilon$.

(i) (8 分) Show that all continuous functions on $[a,b]$ are Darboux integrable.

(ii) (7 分) Show that all monotone functions on $[a,b]$ are Darboux integrable.

試題隨卷繳回