※ 注意：請於試卷內之「非選擇題作答區」依序作答，並應註明作答之部份及題號。

請詳細列出計算及推導過程，否則不予計分。題目前括號( )內之數字為該題配分。

1. (25 points) Let $X_1, \cdots, X_n$ be i.i.d. from the Normal distribution $\mathrm{N}(\mu_1, \sigma_1^2)$, and let $Y_1, \cdots, Y_m$ be i.i.d. form the Normal distribution $\mathrm{N}(\mu_2, \sigma_2^2)$. Assume that $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_m)$ are independent.

   (1) Construct a two-sided $100 \times (1-\alpha)\%$ confidence interval for the ratio of two variances from normal distributions, σ_1^2/σ_2^2.

   (2) If we assume that $\mu_1 = \mu_2 = \mu$ and σ_1^2 and σ_2^2 are known, please find constants c and d such that the estimator $\hat{\mu} = c\bar{X} + d\bar{Y}$ is unbiased for the population mean μ with the minimum variance.

   (3) Find the UMVUE (uniformly minimum variance unbiased estimator) of $(\sigma_1/\sigma_2)^r$, where $r > 0$ and $r < m$.

   (4) Assume that $\sigma_1^2 = \sigma_2^2 = \sigma^2 > 0$. Find the UMVUE's of σ^2 and $(\mu_1 - \mu_2)/\sigma$.

2. (10 points) Let $X_1, \cdots, X_n$ be a random sample from the following discrete distribution:

$$P(X_1 = 1) = \frac{2(1-\theta)}{2-\theta}, \quad P(X_1 = 2) = \frac{\theta}{2-\theta}$$

   where $\theta \in (0,1)$ is unknown. Obtain an estimator of θ using the method of moments and its asymptotic distribution by the central limit theorem and δ −method.

3. (15 points) Let $X_1, \cdots, X_n$ be a random sample from the uniform distribution on the interval $(\theta,\ \theta + |\theta|)$. Find the MLE (maximum likelihood estimate) of θ when

   (1) $\theta \in (0, \infty)$;

   (2) $\theta \in (-\infty, 0)$;

   (3) $\theta \in \mathbb{R}, \theta \neq 0$.

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4. (20 points) Given two independent samples from exponential distribution as $X_1, \dots, X_n \sim f(x|\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$ and $Y_1, \dots, Y_m \sim f(y|\mu) = \frac{1}{\mu} e^{-\frac{y}{\mu}}$. Consider the hypothesis $H_0: \lambda = \mu$ vs. $H_1: \lambda \neq \mu$.

(1) Find the maximum likelihood estimate (MLE) of λ under H_0.

(2) Show that the likelihood ratio test (LRT) can be based on the test statistic

$$T = \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j}.$$

(3) Find the null distribution of T.

5. (30 points) Consider an experiment where in n pairs of patients, one is given the therapy and the other is given a placebo. For the i-th pair, denote $X_i = 1$ if patient receiving the therapy has better recovery, and $X_i = 0$ otherwise. Define $p = (X_i = 1)$ to be the probability that the therapy is effective and $X = \sum_{i=1}^{n} X_i$.

(1) Show that $\mathrm{p} = 0.5$ when the therapy has no effect.

(2) Consider the hypothesis $H_0: p = 0.5$ vs. $H_1: p \neq 0.5$. Show that LRT rejects H_0 for large value of $\left|X - \frac{n}{2}\right|$, i.e., if $\left|X - \frac{n}{2}\right| > k$ for some $k > 0$.

(3) Based on the distribution of X, how to determine k so that the type-I error of LRT is less than α.

(4) Given $n = 10$ and $k = 4$. What is the type-I error of LRT? What is the power of LRT under $p = 0.8$?

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