國立臺灣大學 109 學年度碩士班招生考試試題

(1) Let $\gamma(s)$ be a smooth curve in $\mathbb{R}^3$, parametrized by arc-length. Suppose the $\gamma''(s)$ is nowhere zero. Denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the Frenet frame. Consider $F(s,u) = \gamma(s) + u\mathbf{B}(s) \in \mathbb{R}^3$.

(a) [8%] For any s_0, show that there exists an $\varepsilon > 0$ such that the image of $(s_0 - \varepsilon, s_0 + \varepsilon) \times (-\varepsilon, \varepsilon)$ under F is a regular surface.

From now on, assume the image of F is a regular surface.

(b) [12%] Compute its first fundamental form, and the Christoffel symbols.

(c) [10%] Show that $\gamma(s)$ is a geodesic on the surface defined above.

(2) [20%] If two oriented, regular surfaces, S_1 and S_2, intersect along a regular curve C. Prove the following relation:

$$\kappa^2 \sin^2\theta = (\lambda_1)^2 + (\lambda_2)^2 - 2\lambda_1\lambda_2\cos\theta$$

where

- κ is the curvature of C, as a curve in $\mathbb{R}^3$;
- λ_j is the normal curvature of C on S_j, for $j = 1, 2$;
- θ is the angle made up by the normal vectors of S_1 and S_2.

(3) [20%] Let Σ be an oriented, minimal, regular surface in $\mathbb{R}^3$. Prove that the Gauss map from Σ to $\mathbf{S}^2$ is a conformal map (on where $K \neq 0$).

(4) Let S be the surface $\{\,(u, v, \frac{1}{2}(u^2 - v^2)) \mid (u,v) \in \mathbb{R}^2\,\}$.

(a) [10%] Compute the Gaussian curvature of S.

(b) [20%] Does S contain a smooth, simple closed geodesic? Give your reason. (Hint: S is diffeomorphic to $\mathbb{R}^2$. Due to the Jordan curve theorem, any simple closed curve must bound a topological disk.)

試題隨卷繳回