國立臺灣大學 111 學年度碩士班招生考試試題

1. (15 %) Compute the improper integral $\int_1^\infty x^{-p}dx$ for $p \leq 1$.

2. (10 %) Compute the limit of the bivariate function $f(x,y) = \frac{5x^2y}{x^2+y^2}$ as (x,y) approaches $(0,0)$.

3. Let $\phi(x) = e^{-x^2/2}$.

   (a) (8 %) Show that $\int_{-\infty}^\infty \phi(x)dx = \sqrt{2\pi}$.

   (b) (9 %) Show that $\int_{-\infty}^\infty x^{k+1}\phi(x)dx = k\int_{-\infty}^\infty x^{k-1}\phi(x)dx$ for $k \in \mathbb{N} = \{1,2,3,\ldots,\}$.

   (c) (8 %) Evaluate $\int_{-\infty}^\infty x^k\phi(x)dx$ where $k \in \mathbb{N} = \{1,2,3,\ldots,\}$.

4. (20 %) Let I_p be a $p \times p$ identity matrix, 1_p be a $p \times 1$ vector of one's, and ρ be any constant with $|\rho| < 1$. Find the eigenvalues and eigenvectors of $P = (1-\rho)I_p + \rho 1_p 1_p^\top$.

5. (15 %) Let A be a $p \times p$ square matrix with $A^2 = A$ and $\text{rank}(A) = k < p$. Find the eigenvalues of A and their algebraic multiplicities.

6. (15 %) Let $x \in \mathbb{R}$ and define an $(n+1) \times (n+1)$ matrix

$$T_n = \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \\ x & 1 & x & \cdots & x^{n-1} \\ x^2 & x & 1 & \cdots & x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^n & x^{n-1} & x^{n-2} & \cdots & 1 \end{bmatrix}.$$

   Compute $\det(T_n)$.

試題隨卷繳回