1. (15 points) Consider a 13×13 grid in the plane, such as $\{(x, y) \in \mathbb{Z}^2 | 0 \leq x, y \leq 12\}$. Suppose that each grid point is colored one of red, blue, and green. Prove that there exists a rectangle with all four corners having the same color. If you cannot solve this problem, proving the same fact on a 100×100 grid gives good partial score.

2. (15 points) Solve $T(n)$ for all n that are powers of 2 (show your derivation):

$$\begin{aligned} T(1) &= 2, \\ T(2) &= 3, \\ T(n) &= T(\frac{n}{2}) + 2T(\frac{n}{4}) + \frac{3n}{2}, \text{ for all } n \geq 2. \end{aligned}$$

3. (15 points) Let p be a prime greater than 20. Find all possible values of p^6 mod 182. Prove the correctness of your answer.

4. (35 points) For each of the following statements, determine whether it is true or false. (You must explicitly answer "true" or "false" for each statement separately. No explanation is needed.) You get +5 points for every correct answer and -6 points for every incorrect one. (0 points if you do not answer.)

   (a) $(p \to q) \to r$ and $p \to (q \to r)$ are equivalent.

   (b) $\exists x(P(x) \to Q(x)) \equiv \exists x P(x) \to \exists x Q(x)$.

   (c) In propositional logic, a functionally complete set must have at least two operators.

   (d) If 2^S is uncountable, then S must be infinite and countable.

   (e) The set $S = \{a + b\pi | a, b \in \mathbb{Q}\}$ is countable.

   (f) The symmetric closure of a partial ordering must be an equivalence relation.

   (g) The set $\{(f_1(n), f_2(n)) \mid f_1(n) \leq f_2(n)\}$ is a partial ordering on the set of all positive functions $f : \mathbb{N} \to \mathbb{R}^+$.

5. Let G be a planar graph. Every node in G has degree exactly 5.

   (a) (5 points) Is it possible that G has exactly 24 edges? Prove your answer.

   (b) (10 points) Is it possible that G has exactly 25 edges? Prove your answer.

6. (5 points) Let G be a graph in which every node has degree 5. Prove that the chromatic number of G is at least 5 or find a counterexample.

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