國立臺灣大學 112 學年度碩士班招生考試試題

1. (15 points)
Given two sets $A, B \subset \mathbb{R}^n$ and define their distance by

$$d(A,B) = \inf\{|x-y| : x \in A, y \in B\}.$$

(1) Prove that if one of the sets A, B contains a point in the closure of the other one, then $d(A,B) = 0$.
(2) If A is compact, B is closed and $A \cap B = \emptyset$, then $d(A,B) > 0$.
(3) If A, B are both closed and disjoint. Prove or disprove $d(A,B) > 0$.

2. (15 points)
Let $S \subset \mathbb{R}^n$ be a compact set and $\{U_i\}_{i=1}^{\infty}$ be an open covering of S. Prove that there exists a number $r > 0$ such that for any two points $x, y \in S$ and $|x-y| < r$, then x, y are both contained in U_i for some i.

3. (15 points)
Let $f(x) = x^2 - x + \frac{1}{6}$ on $[0,1]$ and extend to $\mathbb{R}$ with period 1.
(1) Prove that $f(x) = \frac{1}{2\pi^2}\sum_{n\neq 0}\frac{1}{n^2}e^{2\pi i n x}$.
(2) Prove that for any integer $M \geq 1$, we have $\sum_{m=1}^{M}(1-\frac{m}{M+1})f(mx) \geq \frac{-1}{2}$.

4. (10 points)
Let S be the surface $\{(x,y,z) : x^2+y^2 = z^2; 0 \leq z \leq 1\}$ and be oriented so that the normal points upward. Let $F(x,y,z) = (x^2, yz, y)$. Compute $\int\int_S F \cdot n dA$.

5. (15 points)
Suppose $f(x)$ is defined on $[1,\infty)$ and is positive. Assume that f is monotone and $f(x) = o(\int_1^x f(t)dt)$ when $x \to \infty$ (little o). Prove that $f(x)^{1/2} = o(\int_1^x f(t)^{1/2}dt)$ when $x \to \infty$.

6. (15 points)
Suppose f and g are all polynomials of degree $\leq d$. If for some $C, c > 0$ so that we have $|f(x)-g(x)| \leq C|x|^{d+1}$ for any $|x| \leq c$. Do we have $f = g$? Justify your answer.

7. (15 points)
Assume that $a_n > -1$ for all n. Prove the following:
(1) $\sum a_n$ converges absolutely if and only if $\sum \log(1+a_n)$ converges absolutely.
(2) Let $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. Then $\sum a_n$ converges conditionally but $\sum \log(1+a_n)$ diverges.

試題隨卷繳回