國立臺灣大學113學年度碩士班招生考試試題

題號：307

科目：工程數學(C)

複選題(100%): 共20題，每題5分，全部選項答對方能得分，錯一個選項(含) 以上或空白不予計分。考生應作答於答案卡，未作答於答案卡者，不予計分。

1. Find the dimension of the subspace H

$$H = \left\{ \begin{bmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ 2c - a \\ 7b + 6c - 3a \end{bmatrix} : a, b, c \text{ in } \mathbb{R} \right\}$$

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

2. Let W be the subspace spanned by the $\mathbf{u}_1$ and $\mathbf{u}_2$, and find the orthogonal projection of $\mathbf{y}$ onto W, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(A) $\begin{bmatrix} 1/3 \\ -2 \\ -1/4 \end{bmatrix}$ (B) $\begin{bmatrix} 1/4 \\ -5 \\ -1/3 \end{bmatrix}$ (C) $\begin{bmatrix} 1/7 \\ -2 \\ -3/4 \end{bmatrix}$ (D) $\begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$ (E) $\begin{bmatrix} 1/5 \\ -3 \\ -1/4 \end{bmatrix}$

3. The mapping is defined by $T(a_0+a_1t+a_2t^2) = 3a_0+(5a_0-2a_1)t+(4a_1+a_2)t^2$.

If the mapping is linear, find the matrix representation of T relative to the basis $\beta = \{1, t, t^2\}$.

(A) $\begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 & 3 \\ 6 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ (E) $\begin{bmatrix} 2 & 0 & 0 \\ 6 & -2 & 0 \\ 0 & 3 & 3 \end{bmatrix}$

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4. Let $\beta = \{b_1, b_2\} = \left\{\begin{bmatrix}7\\5\end{bmatrix}, \begin{bmatrix}-3\\-1\end{bmatrix}\right\}$ and $C = \{c_1, c_2\} = \left\{\begin{bmatrix}1\\-5\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\right\}$ be bases for $\mathbb{R}^2$, and find the change-of-coordinates matrix from β to C.

(A) $\begin{bmatrix}-2 & 2\\3 & 4\end{bmatrix}$ (B) $\begin{bmatrix}3 & 1\\4 & 2\end{bmatrix}$ (C) $\begin{bmatrix}-3 & 8\\6 & 7\end{bmatrix}$ (D) $\begin{bmatrix}4 & 3\\7 & 2\end{bmatrix}$ (E) $\begin{bmatrix}-3 & 1\\-5 & 2\end{bmatrix}$

5. Find an invertible matrix P such that $A = PDP^{-1}$, where D is a diagonal matrix and

$$A = \begin{bmatrix}1 & 3 & 3\\-3 & -5 & -3\\3 & 3 & 1\end{bmatrix}.$$

(A) $\begin{bmatrix}1 & -2 & -2\\1 & -1 & 0\\0 & 3 & 1\end{bmatrix}$ (B) $\begin{bmatrix}1 & -1 & -1\\-1 & 1 & 0\\1 & 0 & 1\end{bmatrix}$ (C) $\begin{bmatrix}1 & -2 & -2\\1 & -3 & 0\\0 & 0 & 1\end{bmatrix}$

(D) $\begin{bmatrix}1 & 2 & 4\\-1 & -3 & 0\\1 & 0 & 2\end{bmatrix}$ (E) $\begin{bmatrix}1 & 2 & 4\\-1 & -4 & 0\\0 & 0 & 2\end{bmatrix}$

6. Let $Q(\mathbf{x}) = x_1^2 - 8x_1x_2 - 5x_2^2$, where x represents a variable vector in $\mathbb{R}^2$. A change of variable is defined as $\mathbf{x} = \boldsymbol{P}\mathbf{y}$, where $\boldsymbol{P}$ is an invertible matrix, and $\mathbf{y}$ is a new variable vector in $\mathbb{R}^2$. Find this matrix $\boldsymbol{P}$ such that $Q(\mathbf{y}) = 3y_1^2 - 7y_2^2$ after making the change of variable.

(A) $\begin{bmatrix}2/\sqrt{3} & -2/\sqrt{5}\\1/\sqrt{5} & 2/\sqrt{3}\end{bmatrix}$ (B) $\begin{bmatrix}2/\sqrt{3} & -2/\sqrt{5}\\1/\sqrt{5} & 3/\sqrt{3}\end{bmatrix}$ (C) $\begin{bmatrix}-3/\sqrt{7} & 1/\sqrt{7}\\1/\sqrt{7} & 3/\sqrt{7}\end{bmatrix}$

(D) $\begin{bmatrix}2/\sqrt{3} & 1/\sqrt{3}\\-1/\sqrt{3} & 2/\sqrt{3}\end{bmatrix}$ (E) $\begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\-1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}$

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7. Find the dimension of the Null space of the following matrix

$$\begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

8. Given a n x n matrix P, which of following statement are false? (multiple answer)
(A) If P is an invertible matrix, the number 0 is an eigenvalue of A.
(B) If P is an invertible matrix, the determinant of P is not zero.
(C) If P is triangular, then the determinant of P is the product of the entries on the diagonal of P.
(D) If P is an invertible matrix, the dimension of Null space of P is not zero.
(E) If P is not an invertible matrix, the columns of P form a linearly independent set.

9. which of following statement are true? (multiple answer)
(A) Given an m x n matrix A, the rank of A plus dimension of Null space of A is equal to m.
(B) The rank of matrix A is the dimension of the row space of A.
(C) Let A is m x n and **b** is $\mathbb{R}^m$. Considering a least-squares problem A**x**=**b** if the columns of A are linearly independent, the solution of this problem $\hat{\mathbf{x}}$ is given by $(A^TA)^{-1}A^T\mathbf{b}$, where A^T is the transpose of A.
(D) An eigenvector of an n x n matrix is a nonzero vector x such that A**x**=λ**x** for some scalar λ.
(E) Given four n x n matrices A, B, C, and D, the tr(ABCD) is equal to the tr(CDAB), where tr( ) means the trace of the matrix.

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10. The vectorization of a matrix is a linear transformation converting the matrix into a vector. For example, given a 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the vectorization of A is $\text{vec}(A) = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix}$. Also, If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is another $pm \times qn$ matrix, and the A^T is the transpose of A. Now for matrices A, B, and C of dimensions $k \times l$, $l \times m$, and $m \times n$, which of the following is equal to the vec(ABC)?

(A) $(A^T \otimes B)\text{vec}(C)$

(B) $(A \otimes B^T)\text{vec}(C)$

(C) $(C \otimes A^T)\text{vec}(B)$

(D) $(C^T \otimes A)\text{vec}(B)$

(E) $(C^T \otimes B)\text{vec}(A)$

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11. Which of the following statements are NOT true?
(A) The function $y = 0$ consistently serves as a solution for a linear homogeneous ODE.
(B) A first-order ODE can be both exact and linear separable.
(C) The ODE $y' = e^{8x+7y}$ is not separable.
(D) For any ODE, the sum of any two solutions constitutes a valid solution as well.
(E) $y\frac{d^2y}{dx^2} = 1$ is a linear ODE.

12. Which of the following statements are NOT true?
(A) The general solution to $\frac{dy}{dx} + \frac{y}{x} = x^2$ is $y = \frac{x^3}{3} + cx$.
(B) The general solution to $\frac{dy}{dx} = \frac{3}{\sqrt{2x+11}}$ is $y = 3\sqrt{2x+11} + c$.
(C) The general solution to $\frac{dy}{dx} + xy = 2x$ is $y = 2 + ce^{0.5x^2}$.
(D) The general solution to $2x^2\frac{dy}{dx} - 2xy = 3$ is $y = -\frac{3}{4x} + \frac{c}{x^2}$.
(E) The general solution to $3x\frac{dy}{dx} + y = 24x$ is $y = 6x + cx^{-1/3}$.

13. Let $y_p = A\sin 2t + B\cos 2t$ be the steady-state solution of the ODE $y'' + 4y' + 20y = 2\cos 2t$. Which of the following equations are correct?
(A) $16A - 8B = 0$ (B) $8A + 16B = 0$ (C) $A = 0.01$
(D) $A = 0.05$ (E) $B = 0.02$

14. For the differential equation $y' + 7y = 10\delta(t-2)$ with the initial condition $y(0) = 10$, the solution includes:
(A) $y = 7e^{-10t}$ for $t < 2$
(B) $y = 10e^{-7t}$ for $t < 2$
(C) $y = 20e^{-7t}$ for $t \geq 2$
(D) $y = 7e^{-10t} + 7e^{-10(t-2)}$ for $t \geq 2$
(E) $y = 10e^{-7t} + 10e^{-7(t-2)}$ for $t \geq 2$

15. Let $y = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots$ be the Taylor polynomial approximation for the initial value problem

$$y' + (\sin t)y = 0,\ y(0) = 1.$$

Please select the correct equations regarding a_i, $i = \{0, 1, \cdots, 4\}$:
(A) $a_0 + a_1 + a_2 + a_3 + a_4 = 1$ (B) $a_0 - a_1 = 1$ (C) $a_2a_3 + a_4 = 0$
(D) $a_2 = -3a_4$ (E) $a_3 = 2a_4$

16. Solve the initial value problem

$$y' - \frac{3y}{t+1} = (t+1)^2,\ y(0) = 2.$$

Please select the correct solutions:
(A) $y(1) = \ln 2 + 2$
(B) $y(1) = 8\ln 2 + 8$
(C) $y(2) = 27\ln 3 + 54$
(D) $y(2) = 27\ln 3 + 72$
(E) $y(3) = 64\ln 4 + 192$

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17. Consider the following second-order differential equation

$$y'' + 6y' + 34y = 2e^{-t},\ y(0) = y'(0) = 0.$$

Which of the following terms are NOT included in the solution $y(t)$?

(A) $-\frac{1}{10}e^{-t}$ (B) $-\frac{1}{10}e^{-3t}\cos(4t)$ (C) $-\frac{1}{10}e^{-4t}\cos(3t)$

(D) $-\frac{1}{20}e^{-3t}\sin(4t)$ (E) $-\frac{1}{20}e^{-4t}\sin(3t)$

18. Consider the system $x' = -2x + 4xy$, $y' = -2y + 2x^2$. Which of the following statements are NOT true?

(A) The equilibrium solutions of this system include $(0,0)$ and $(\pm 1/\sqrt{2}, 1/2)$.
(B) The system is asymptotically stable at $(0,0)$.
(C) The system is asymptotically stable at $(\pm 1/\sqrt{2}, 1/2)$.
(D) This system can have a non-constant, periodic orbit.
(E) $x(t)$ converges to 0 if $x(0) = 0.1$ and $y(0) = 0.1$.

19. For the differential equation $x^2y''(x) + xy'(x) - 25y(x) = 0$, please find the most general solution $y(x)$.

(A) $y(x) = C_1x + C_2x^{-25}$
(B) $y(x) = C_1x^5 + C_2x^{-5}$
(C) $y(x) = C_1x^{25} + C_2x^{-1}$
(D) $y(x) = C_1x^{-5} + C_2x^{-20}$
(E) None of above.

20. Compute the convolution $f(t) = 2t * (\sin t - e^t)$. Which of the following terms are included in $f(t)$?

(A) $4t$ (B) $-\sin t$ (C) $2\cos t$ (D) $-e^t$ (E) 2

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