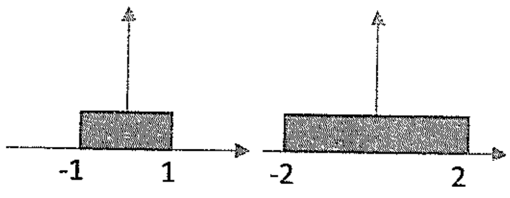
第 1 題至第 5 題為單選題，每題 10 分，
請用 2B 鉛筆作答於答案卡，並先詳閱答案卡上之「畫記說明」。

1. X and Y are independent continuous random variables with probability density functions (PDFs) shown below. What is the probability of $P(-1<X+Y<1)$? (A) 1/4 (B) 2/3 (C) 3/4 (D) 5/8 (E) none of the above.



2. X and Y are random variables with a joint PDF: $P(X=x, Y=y)=e^{-y}$ for $x>=0$ and $y>=x$. What are the probability densities of $P(X=0.5|Y=3)$, $P(X=2|Y=3)$, and $P(X=2.5|Y=3)$, respectively? (A) 1/3, 1/3, and 0 (B) 0, 1/3, and 1/3 (C) 1/3, 0, and 1/3 (D) 1/3, 1/3, and 1/3 (E) none of the above.

3. X is a continuous random variable as given in Problem 1. Let $Y=-2X^2$. Please derive the cumulative distribution function (CDF) of Y --- F(Y) --- and calculate F(-3), F(-1) and F(1), respectively. (A) 0, $1-(1/2)^{1/2}$ and 1 (B) 0, $2^{1/2}-1$, and 1 (C) 1/2, $2^{1/2}/2$, and 1 (D) 0, $2^{1/2}-1$, and 0 (E) none of the above.

4. Bob goes to a temple for some advices from God. Before he can make a request, he needs to get an approval first by tossing a pair of fair crescent-shaped "coins." If both coins are "head", it is called a smile-cup and he can try again. If both are "tail", it is called a no-cup, and he has to stop and leave the temple. If one of the coins is "head" and the other is "tail", it is called an approval. Based on this ritual, what is the probability that he gets an approval from God? (A) 1/2 (B) 5/8 (C) 2/3 (D) 3/4 (E) none of the above.

5. Based on Problem 4, how many tosses on average Bob made before he leaves the temple (either he is rejected or gets an approval)? (A) 1 (B) 4/3 (C) 3/2 (D) 2 (E) none of the above.

見背面

※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

6. (21%) Let V be an n-dimensional vector space associated with the field of the real number set $\mathcal{R}$. Consider a linear operator $T : V \to V$. Suppose there exist n vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in V$ that are linearly independent, each of them being an eigenvector of T. In other words, for any $k \in \{1, 2, ..., n\}$,

$$T(\mathbf{x}_k) = \lambda_k \mathbf{x}_k,$$

for some $\lambda_k \in \mathcal{R}$.

(a) (7%) Let $\mathbf{u} = \mathbf{x}_1 + 2\mathbf{x}_2 + \cdots n\mathbf{x}_n$. Find $T(\mathbf{u})$ and express it in terms of λ_k's and $\mathbf{x}_k$'s. Justify your answer.

(b) (7%) Prove or disprove that T is **diagonalizable**, that is, there exist an ordered basis $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ and a diagonal matrix D such that for any $\mathbf{v} \in V$, $[T(\mathbf{v})]_{\mathcal{B}} = D[\mathbf{v}]_{\mathcal{B}}$, where $[\cdot]_{\mathcal{B}}$ is coordinate vector notation: $[\mathbf{x}]_{\mathcal{B}} = [c_1, c_2, ..., c_n]^T \in \mathcal{R}^n$ where c_k's are the unique set of scalars that satisfies $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$ for any $\mathbf{x} \in V$.

(c) (7%) Find the necessary and sufficient condition of the scalars λ_k's such that T is an invertible operation (i.e., there exists $U : V \to V$ such that $T(U(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for any $\mathbf{x} \in V$).

7. (9%) Consider the vector space $\mathcal{R}^4$ over the field $\mathcal{R}$ with standard addition

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 & u_2 + v_2 & u_3 + v_3 & u_4 + v_4 \end{bmatrix}^T$$

and standard scalar multiplication

$$c \cdot \mathbf{u} = \begin{bmatrix} cu_1 & cu_2 & cu_3 & cu_4 \end{bmatrix}^T$$

for any $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T, \mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^T \in \mathcal{R}^4$ and $c \in \mathcal{R}$. Suppose we define

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 + u_3v_3.$$

Is $\langle \cdot, \cdot \rangle$ an inner product of the vector space $\mathcal{R}^4$? Why?

8. (20%) Consider an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ and let its reduced row echelon form be $R = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_n \end{bmatrix}$. Then, there exists an invertible matrix P of size $m \times m$ such that $PA = R$.

(a) (10%) Suppose a subset of column vectors of A, $\{\mathbf{a}_{p_1}, \mathbf{a}_{p_2}, \cdots \mathbf{a}_{p_k}\}$, is linearly independent. Here k is an integer, $k \leq n$, and $1 \leq p_1 < p_2 < \cdots < p_k \leq n$. Show that $\{\mathbf{r}_{p_1}, \mathbf{r}_{p_2}, \cdots \mathbf{r}_{p_k}\}$ is also linearly independent.

(b) (10%) Show that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathcal{R}^m$ if and only if rank $A = m$ ("if": 5%; "only if": 5%).

試題隨卷繳回