※注意：請於試卷上「非選擇題作答區」標明題號並依序作答。

1. (15 %) Suppose that f is a continuous function on a closed interval $[a,b]$. Show that f is uniformly continuous on $[a,b]$.

2. (20 %) Let $\mathbb{N}$ be the set of all natural numbers and $n \in \mathbb{N}$.

   (a) Consider a sequence $\{a_n\}_{n=1}^{\infty}$ defined by $\begin{cases} a_1 = & \sqrt{2} \\ a_{n+1} = & \sqrt{2+a_n} \end{cases}$ for $n \geq 1$. Show that $\lim_{n\to\infty} a_n$ exists and evaluate the limit.

   (b) Prove the relation $\lim\limits_{n\to\infty} \dfrac{1}{n^{k+1}} \sum\limits_{i=1}^{n} i^k = \dfrac{1}{k+1}$ for any nonnegative integer k.

3. (15 %) (a) If $y' = ay$, show that $y = ce^{ax}$ for some c. (b) If $f(x+y) = f(x)f(y)$ and f is differentiable, show that either $f(x) \equiv 0$ or $f(x) = e^{ax}$ for some a.

4. (15 %) Show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

5. (15 %) Calculate

$$\iint_S z\,dx \wedge dy - x\,dy \wedge dz,$$

   where S is the spherical cap $x^2+y^2+z^2=1$, $x > 1/2$, oriented positively with respect to the normal pointing to infinity.

6. (20 %) Prove that the function $f(x,y) = e^{-y^2+2xy}$ can be expanded in a series of the form

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n,$$

   that converges for all values of x and y and that the polynomials $H_n(x)$ satisfy

   (a) $H_n(x)$ is a polynomial of degree n

   (b) $H_n'(x) = 2nH_{n-1}(x)$

   (c) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$

   (d) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

試題隨卷繳回