Throughout the Exam, $\mathbb{R}$ denote the set of all real numbers, $\mathbb{C}$ denote the set of all complex numbers, and i denote imaginary unit. We use 'T' to denote the *transpose* of a vector or matrix. We use boldface symbol, e.g. **b**, to denote a column vector.

The column space of a matrix A consists of all linear combinations of the columns of A. The nullspace of A contains all solutions $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$. The row space and the left nullspace of A are the column space and the null space of the transpose matrix A^T, respectively.

1. (25%) At noon on a particular day, an experiment is conducted to record new call attemps at a telephone switchboard. Random variable X denotes the arrival time of the first call, as measured by the number of seconds after noon, and random variable Y denotes the arrival time of the second call. It is found that X and Y are continuous random variables with joint PDF as follows:

$$f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x < y, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ calls/second is a constant.

(a) (6%) Derive the marginal PDFs of the arrival times of the first and second calls respectively.

(b) (4%) Derive the expected values of the arrival times of the first and second calls respectively.

(c) (5%) Let W be the random variable denoting the time difference between the arrivals of the second call and the first call. Derive the PDF of W.

(d) (5%) It is observed that the second call arrives at time $y > 0$, but the arrival time of the first call is unknown. Derive the PDF of the arrival time of the first call under this observation.

(e) (5%) It is observed that the first call arrives at time $x > 0$. At time $x + \tau$, $\tau > 0$, it is also observed that the second call still does not arrive yet. Derive the PDF of the arrival time of the second call under this observation.

2. (7%) X_1 and X_2 are identically distributed random variables with expected value μ and variance σ^2. Let $Y = X_1 + kX_2$ be another random variable, where k is a constant. Find the range of k such that the variance of Y is greater than or equal to σ^2.

3. (18%) Alice participates in a tournament, in which a series of games are played indefinitely until she loses. The scoring rule of the game is that the winner is granted 1 point, the loser is deducted 1 points, and a tie grants both players 0 points. Each game is played independently, and it is known that in each game the probability for Alice to win is 0.5 and to lose is 0.4. Let X be the total number of points Alice earns in the tournament.

(a) (8%) Derive the MGF $\phi_X(s)$ of X.

(b) (5%) Derive the variance of X.

(c) (5%) Derive an upper bound of the probability that X is greater than or equal to 9 times of its expected value using the Chebyshev Inequality.

4. (5%) Answer the following questions with "True" or "False".

(a) The vectors $[1,3,2]$ and $[2,1,3]$ and $[3,2,1]$ are dependent.

(b) The vectors $[1,-3,2]$ and $[2,1,-3]$ and $[-3,2,1]$ are dependent.

(c) If the row space equals the column space, then $A^T = A$.

(d) If $A^T = -A$, then the row space of A equals the column space.

(e) A and A^T have the same number of pivots.

(Getting 5 points if all answers are correct. Otherwise, 0 point.)

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5. (5%) Suppose columns of a matrix A are n vectors in $\mathbb{R}^m$. Answer the following questions with "True" or "False".
   (a) Let A be a real matrix. Then, A^TA is invertible if and only if the columns of A are linearly independent
   (b) The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if A has a pivot position in every row.
   (c) Let $n = 3$ and $m = 5$. Then $A\mathbf{x} = \mathbf{b}$ is not necessarily solvable.
   (d) There exists a 3×3 diagonal matrix A such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.
   (e) If $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, then the null space of A is span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)
6. (5%) Answer the following questions.
   (a) $A^T\mathbf{y} = \mathbf{b}$ is solvable when $\mathbf{b}$ is in which of the four subspaces of A? ______
   (b) As above, the solution $\mathbf{y}$ is unique when the ______ contains only the zero vector.
   (c) The dimension of the subspace of $n \times n$ symmetric matrices is ______.
   (d) Suppose the kernel of an 5×3 matrix A is one dimensional. What is the dimension of A?
   (e) (True or False) Suppose A, B and C are 2×2 matrices such that A commute with B and B commute with C. Then, A commutes with C.
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)
7. (5%) Suppose A is the sum of two matrices of rank one: $A = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$. Answer the following questions.
   (a) Which vectors span the column space of A?
   (b) Which vectors span the row space of A?
   (c) The rank is less than 2 if ______ or if ______.
   (d) What is the rank of A if $\mathbf{u} = \mathbf{z} = [1, 0, 0]^T$ and $\mathbf{v} = \mathbf{w} = [0, 0, 1]^T$?
   (Getting 5 points if all answers are correct. Otherwise, 0 point.)
8. (5%) The matrix
$$\begin{bmatrix} 51 & -12 & -21 \\ 60 & -40 & -28 \\ 57 & -68 & 1 \end{bmatrix}$$
   has eigenvalues −48 and 24. What is the third eigenvalue?
9. (10%) Let
$$\begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}.$$
   Calculate the power A^{2024}.
10. (15%) Find an orthonormal basis of polynomials on real lines with degree at most 2, where the inner product is given by $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\mathrm{d}x$.