- Any device with computer algebra system is prohibited during the exam.
- Each answer must be clearly labeled on the answer sheet.
- You need to provide arguments for your answers.

1. (a) (5 pts) Assume that $f'(a) > 0$. Use the definition of derivative to prove that there is some $\epsilon > 0$ such that $f(x) > f(a)$ for all $x \in (a, a+\epsilon)$.

   (b) (15 pts) Suppose that $f(x)$ is differentiable on an open interval containing $[a,b]$ with $f'(a) \neq f'(b)$ and $m$ is a constant between $f'(a)$ and $f'(b)$. Prove that there is some $c \in (a,b)$ such that $f'(c) = m$.

2. Let $\{f_n(x)\}$ be a sequence of continuous functions defined on $[0,1]$. Assume that $f_n(x)$ converges to $f(x)$ uniformly on $[0,1]$.

   (a) (9 pts) Prove that $f(x)$ is continuous on $[0,1]$ and there is some $M > 0$ such that $|f_n(x)| < M$ for all $x \in [0,1]$ and positive integer $n$.

   (b) (6 pts) Prove or disprove $\lim\limits_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx$.

3. (a) (9 pts) Suppose that $1 < b < 2$ is a fixed constant and

$$a_n = (1-\frac{b}{2})(1-\frac{b}{3})\cdots(1-\frac{b}{n}) \text{ , for } n \geq 2.$$

   Show that there are constants $0 < m < M$ such that $m < n^b a_n < M$ for all $n \geq 2$.

   (b) (6 pts) Show that for $k > 0$, the series $\sum_{n=1}^{\infty} \binom{k}{n}$ converges absolutely.

4. Suppose that $f(x,y)$, $g(x,y)$ have continuous second derivatives, and on the level curve $g(x,y) = 0$, $f(x,y)$ obtains a local extreme value at $(1,2)$. Assume that at $(1,2)$, $f_x = 3$, $g_x = -1$, $g_y = \frac{1}{2}$, $f_{xx} = 0$, $f_{xy} = 1$, $f_{yy} = -2$, $g_{xx} = 3$, $g_{xy} = 0$, and $g_{yy} = 1$.

   (a) (2 pts) Find $f_y(1,2)$.

   (b) (7 pts) When restricted to the curve $g(x,y) = 0$, is $f(1,2)$ a local maximum, local minimum, or saddle point?

   (c) (6 pts) Suppose that on the curve $g(x,y) = 10^{-2}$, $f(x,y)$ has a local extreme value at $(x_1, y_1)$ which is close to $(1,2)$. Estimate $f(x_1,y_1) - f(1,2)$ by linear approximation.

5. (15 pts) Suppose that $F(x,y,u,v)$ and $G(x,y,u,v)$ have continuous first partial derivatives and equations $F(x,y,u,v) = 0$ and $G(x,y,u,v) = 0$ can be solved for $x, y$ as functions of $u, v$. Express $\frac{\partial(x,y)}{\partial(u,v)}$ in terms of $\frac{\partial(F,G)}{\partial(u,v)}$ and $\frac{\partial(F,G)}{\partial(x,y)}$, where $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$, $\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$, $\frac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}$ are Jacobian determinants.

6. (20 pts) Let $F(x,y,z) = x^2 + 2y^2 - 3z^2$ and $S$ be the level surface $F(x,y,z) = 2$ between $z = 0$ and $y - \sqrt{3}z + 1 = 0$ with downward orientation. Compute $\iint_S \nabla F \cdot d\mathbf{S}$.

試題隨卷繳回